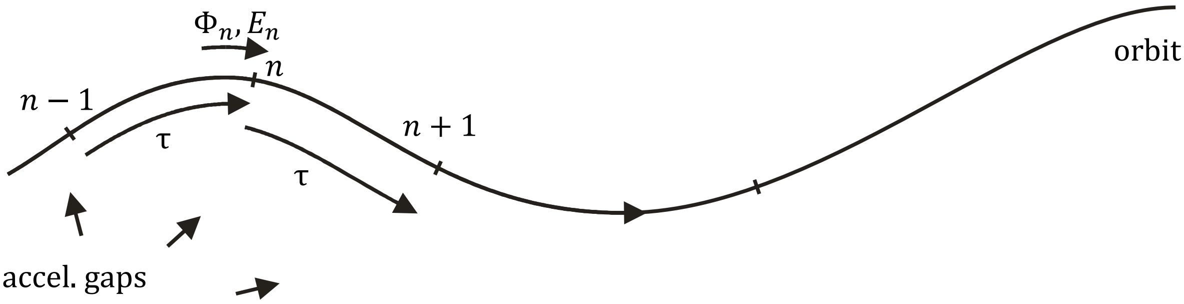
Longitudinal Beam Dynamics

- stability of acceleration
- separatrix & bucket
- synchrotron oscillations
- longitudinal acceptance
- transient time factor
- transverse defocusing in accelerating gap

stability of acceleration



operate at ω_{rf} = const.!

design (perfect) particle:

- τ_s from gap to gap
- energy gain ΔE_s per gap
- arrival at $\phi_i = \phi_s$ $\quad$ $s \hat{=}$ "synchroneous"
- $E = E_{s,n}$ at gap, = kin. + rest energy

real particle:

$\tau = \frac{L}{V}$, $\quad \frac{\partial\tau}{\tau} = \frac{\partial L}{L} - \frac{\partial v}{v} = \frac{\partial L}{L} - \frac{1}{\gamma^2} \cdot \frac{\partial p}{p}$; $\quad$ ∂L from $L = L(\partial p)$ on curved orbits

definition: $\quad \frac{\partial L}{L} := \frac{1}{{\gamma_t}^2} \cdot \frac{\partial p}{p}$ = 0 for Linacs !

$\frac{\partial\tau}{\tau} = \frac{1}{{\gamma_t}^2}\frac{\partial p}{p} - \frac{1}{\gamma^2}\frac{\partial p}{p} := \eta\frac{\partial p}{p}$; $\quad \eta = \frac{1}{{\gamma_t}^2} - \frac{1}{\gamma^2}$; $\quad$ $\gamma > \gamma_t \rightarrow \frac{\partial\tau}{\partial p}$ turns positive!

γ_t = "transition energy"

Linacs: $\gamma_t = \infty$, $\quad \eta < 0$!

Longitudinal Beam Dynamics

Φ_n, E_n: phase and energy at arrival at gap n, maybe $\neq (\Phi_s, E_s)$!

$$\Phi_{n+1} = \Phi_n + \omega_{rf} \cdot \delta\tau_{n+1}$$

$$= \Phi_n + \omega_{rf} \cdot \tau \quad \cdot \frac{\delta\tau_{n+1}}{\tau}$$

$$= \Phi_n + \omega_{rf} \cdot \tau \quad \cdot \eta \cdot \frac{\partial p}{p}|_{n+1} = \Phi_n + \frac{2\pi h \cdot \eta}{\beta^2} \cdot \frac{\partial E}{E_s}|_{n+1}; \qquad \tau \cdot \omega_{rf} = 2\pi h$$

h: "harmonic factor"

$$\delta E_{n+1} = \delta E_n + e \cdot q \cdot U[\cos\Phi_n - \cos\Phi_s]; \quad t = n \cdot \tau; \quad \delta t = (\delta n \equiv 1) \cdot \tau = \tau$$

$$\delta(\frac{d\Phi}{dt}) = \frac{2\pi h \cdot \eta}{\beta^2 \cdot \tau} \cdot \frac{\delta E}{E_s} \qquad (*)$$

$$\frac{d(\delta E)}{dt} = \frac{e \cdot q \cdot U}{\tau}[\cos\Phi - \cos\Phi_s]$$

$$\frac{(*)}{\tau} \to \frac{d^2\Phi}{dt^2} = \ddot{\Phi} = \underline{\frac{2\pi h \cdot \eta}{\tau^2 \cdot \beta^2} \cdot \frac{e \cdot q \cdot U}{E_s}[\cos\Phi - \cos\Phi_s]} \quad \text{oscillation} \qquad (**)$$

$$\Phi = \Phi_s + \delta\Phi, \qquad \delta\Phi \ll 1:$$

$$\cos\Phi - \cos\Phi_s \approx -\delta\Phi \cdot \sin\Phi_s$$

$$\to \delta\ddot{\Phi} \approx -\frac{2\pi h \cdot \eta}{\tau^2 \cdot \beta^2} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot \sin\Phi_s \cdot \delta\Phi \qquad \text{harmonic oscillator}$$

$$\delta\Phi = A \cdot e^{i\omega_{sync} \cdot t}$$

$$\omega_{sync}{}^2 = \frac{\omega_{rf}{}^2 \cdot \eta}{2\pi\beta^2 \cdot h} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot \sin\Phi_s$$

"synchrotron oscillation":

Linac: $\eta < 0 \to 0 > \Phi_s > -\pi$!

integration of $(**)$:

$$\dot{\Phi}\ddot{\Phi} = \frac{2\pi h \cdot \eta}{\tau^2 \cdot \beta^2} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot \dot{\Phi} \cdot [\cos\Phi - \cos\Phi_s] \quad | \int$$

$$\frac{1}{2}\dot{\Phi}^2 = \frac{2\pi h \cdot \eta}{\tau^2 \cdot \beta^2} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot [\sin\Phi - \Phi\cos\Phi_s] \quad | \text{ using } (*)$$

$$\frac{\pi h \cdot \eta}{\beta^2} \cdot \frac{\delta E^2}{E_s} = e \cdot q \cdot U \cdot [\sin\Phi - \Phi\cos\Phi_s] + \text{const.}$$

$$\delta E^2 + \frac{\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta} \cdot [\Phi \cdot \cos\Phi_s - \sin\Phi] = C$$

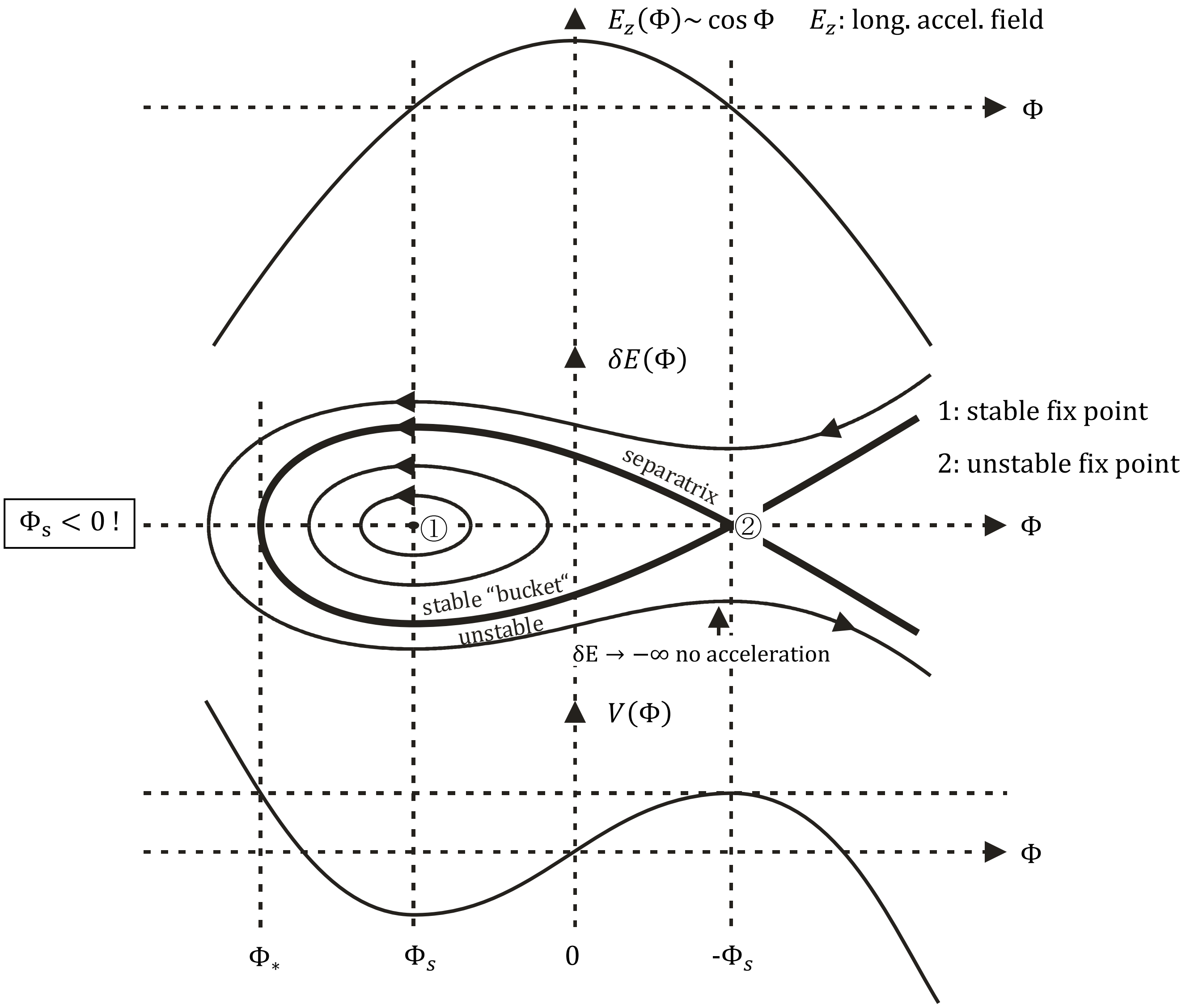
$$\delta E^2 + V(\Phi) = C$$

<u>$\delta\Phi \ll 1$:</u>

$$\Phi \cdot \cos\Phi_s - \sin\Phi \xrightarrow{\text{Taylor to 2}^{\text{nd}}\text{ order in }\delta\Phi} \Phi_s \cdot \cos\Phi_s - \sin\Phi_s + \frac{\delta\Phi^2}{2} \cdot \sin\Phi_s$$

$$\rightarrow \delta E^2 + \frac{\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta} \cdot \left[\Phi_s \cdot \cos\Phi_s - \sin\Phi_s \left[1 - \frac{\delta\Phi^2}{2}\right]\right] = C$$

$\delta\Phi \ll 1$: ellipse equation!



acceleration: $\cos\Phi_s > 0 \rightarrow -\frac{\pi}{2} < \Phi_s < \frac{\pi}{2}$

$\omega_{sync}{}^2 > 0 \rightarrow -\pi < \Phi_s < 0$

$\rightarrow$ stable acceleration for $0 > \Phi_s > -\frac{\pi}{2}$

$|\Phi - \Phi_s| \leq 1 \rightarrow \Phi_* \approx 2\Phi_s$

usually $\Phi_s \approx -30° = -0.52$ rad, approximation $\Phi_* \approx 2\Phi_s$ ok

inside bucket:

particles rotate with $\approx \omega_{sync}$

Acceptance (bucket size)

in phase Φ: $[2\Phi_s; \ \ -\Phi_s]$

$\Phi_s = 0$: maximum acceleration but zero acceptance

$\Phi_s = \pm\frac{\pi}{2}$: zero acceleration but maximum phase acceptance, i.e. $[-\pi, \ \pi]$

$\Phi_s \approx -30°$ compromise

Energy acceptance at $\Phi = \Phi_s$

$\Phi = -\Phi_s \rightarrow \delta E = 0$

$\rightarrow C = \frac{\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta}[\sin\Phi_s - \Phi_s \cdot \cos\Phi]$

$\Phi = \Phi_s \rightarrow \delta E$ at max

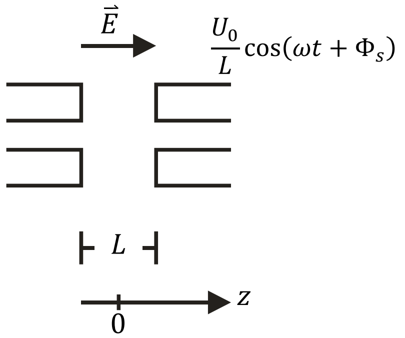
$\rightarrow \boxed{\delta E_{max}{}^2 = \frac{2\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta}[\sin\Phi_s - \Phi_s \cdot \cos\Phi_s]}$

for ions ${}^A X^{q+}$ $\quad q \rightarrow \frac{q}{A}$

$E \rightarrow E_u$

Longitudinal Beam Dynamics

Transient Time Factor



- $z = 0$: $\Phi = \Phi_s$
- L finite $\rightarrow E_z \neq$ const in time
- approximation $\frac{\partial E}{\partial r} = \frac{\partial E}{\partial z} = 0$

$\delta E = \frac{U_0 qe}{L}\cos(\omega t + \Phi_s) \cdot ds$ E: "Energie"

$$\Delta E = \frac{U_0 qe}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos(\omega t + \Phi_s)\, ds$$

$$= \frac{U_0 qe}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left[\omega \cdot \frac{s}{\beta(s)\cdot c} + \Phi_s\right] ds$$

solve with approximation: $\beta(s) \approx \text{const} = \beta$

$$\Delta E = \frac{U_0 qe}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left[\cos\frac{\omega s}{\beta \cdot c} \cdot \cos\Phi_s - \sin\frac{\omega s}{\beta\cdot c} \cdot \sin\Phi_s\right] ds$$

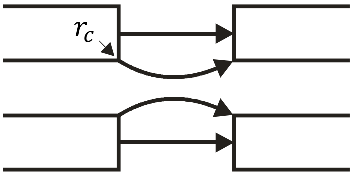
$$= U_0 \cdot qe \cdot \cos\Phi_s \cdot \frac{2\beta c}{\omega L} \cdot \sin\frac{\omega L}{2\beta c} \quad := U_0 \cdot \mathrm{qe} \cdot \cos\Phi_s \cdot \mathrm{T}$$

T corrects for finite gap length, "transient time factor", "TTF", "T"

$$\boxed{T = \frac{\beta\cdot\lambda}{\pi\cdot L} \cdot \sin\frac{\pi\cdot L}{\beta\cdot\lambda}}$$

usual DTLs: $L \approx \frac{\beta\cdot\lambda}{3.5} \rightarrow T \approx 0.8 \cdots 0.9$

<u>including $\vec{E}(\mathrm{r,z})$ + geometry (gap diameter 2a):</u>



$$T = \frac{J_0(2\pi a/\lambda)}{I_0\left(\frac{2\pi a}{\beta\cdot\gamma\cdot\lambda}\right)} \cdot \frac{\sin\frac{\pi L^*}{\beta\lambda}}{\frac{\pi L^*}{\beta\cdot\lambda}}$$

$L^* = L + 0.85 r_c$

Transverse gap defocusing

inside gap: $\vec{E} = -\vec{\nabla}\Psi;\ \vec{\nabla}\cdot\vec{E} = 0$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2\Psi \Longrightarrow \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = 0$$

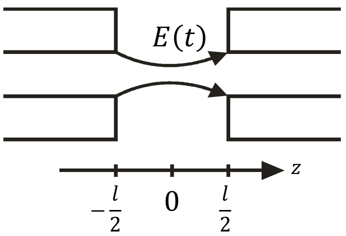
$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -\frac{\partial^2\Psi}{\partial z^2}$$

part inside bucket: focusing in $\Phi \,\widehat{=}\,$ along $z \rightarrow \frac{\partial^2\Psi}{\partial z^2} > 0$

$$\rightarrow \left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right] < 0 \rightarrow \text{transverse defocusing!}$$

without transverse focusing, stable acceleration will cause transverse particle loss!

<u>alternative picture:</u>



$z < 0$: $\vec{E}$ focusing

$z > 0$: $\vec{E}$ defocusing but stronger ($E(t)$!)

} net transverse defocusing !

→ any accelerator needs transverse focusing devices!

<u>focusing strength:</u>

$$d[\delta E] = U_0 \cdot T \cdot q \cdot e \cdot [\cos\Phi - \cos\Phi_s] = -E_0 \cdot \delta s \cdot T \cdot q \cdot e \cdot \sin\Phi_s \cdot \delta\Phi, \qquad \delta s\text{: gap length}$$

$$d[\delta E] = d\left[E\beta^2 \cdot \frac{\partial p}{p}\right] = d[E\beta^2\gamma^2 l']$$

l: ▸ intra-bunch coordinate

▸ part too fast $l' = \frac{\partial l}{\partial z} > 0$

$$\delta\Phi = -\frac{l\cdot 2\pi n}{\beta\cdot\lambda}\Big|\ \frac{2\pi}{n}\text{-mode cavity}$$

$$\Longrightarrow\ dl' = \frac{E_0\cdot\delta s\cdot T\cdot q\cdot e\cdot 2\pi n}{m_0 c^2\beta^3\gamma^3\lambda}\cdot\sin\Phi_s\cdot l$$

$\Phi_s \approx -30°$: $l > 0 \ \rightarrow\ dl' < 0$, focusing

$$dl' \coloneqq q_l \cdot l$$

$q_l \coloneqq$ long. foc. strength < 0

focusing potential $\Psi(l)$ (non-relativistic)

$$\Psi(l) = \frac{1}{2}kl^2$$

$$F = -\frac{\partial \Psi}{\partial l} = -k \cdot l = m_0 \cdot \ddot{l} = m_0 \cdot v^2 \cdot l''$$

$$l'' = -\frac{k}{m_0\, v^2} l, \qquad \delta l' = l'' \cdot \partial s = -\frac{k \cdot \partial s}{m_0 v^2} l$$

$$q_l = -\frac{k \cdot \partial s}{m_0 v^2} \qquad \Longrightarrow k \ \sim q$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\partial^2 \Psi}{\partial z^2} = -\frac{\partial^2 \Psi}{\partial l^2}$$

$$k_x + k_y = 2k_\perp = -k_l \qquad \boxed{\Longrightarrow k_\perp = -\frac{1}{2}k_l \qquad q_\perp = -\frac{1}{2}q_l}$$

Relativistic treatment gives same result